7. Suppose G is a non-cyclic group of order $205 = 5 \ \ 41$. Give, with proof, the number of elements of order 5 in G.

8. Find ALL solutions x in the integers to the simultaneous congruences.

$$x \quad 7 \mod 11$$
$$x \quad 2 \mod 5$$

9.

12. Find, with brief justification, all ring homomorphisms from $\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$.

13. Consider the ring of Gaussian integers $\mathbb{Z}[i]$.

    (a) Prove that if $\alpha = a + bi$ for $a, b \in \mathbb{Z}$ is a Gaussian integer with $N(\alpha) = p$ for p a prime of $\mathbb{Z}$, then α is irreducible.

    (b) List all the units of $\mathbb{Z}[i]$.

    (c) Give an example of a prime number $p \in \mathbb{Z}$ such that p is irreducible in $\mathbb{Z}[i]$. Justify your answer by stating an appropriate result.

14. Let D be a square-free integer, and consider the quadratic number field $\mathbb{Q}(\sqrt{D})$ and its subring of integers $\mathcal{O}$. Let $N : \mathbb{Q}(\sqrt{D}) \to \mathbb{Z}$ denote the field norm map which is multiplicative. The restriction of N to the ring of integers $\mathcal{O}$ will also denoted by N.

    (a) Prove that an element $\alpha \in \mathcal{O}$ is a unit if, and only if, $N(\alpha) = \pm 1$.

    (b) When $D = -3$, the ring of integers is $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\left(\frac{1 + \sqrt{-3}}{2}\right)$. Find a unit in $\mathcal{O} \setminus \mathbb{Z}$.

    (c) Let $D = -5$. Give, with proof, an example of an element $x = a + b\sqrt{-5}$ for $a, b \in \mathbb{Z}$ such that x is irreducible, but x is not prime in $\mathbb{Z}[$